Isomorphisms between the Batalin-Vilkovisky antibracket and the Poisson bracket

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Abstract

One may introduce at least three different Lie algebras in any Lagrangian field theory : (i) the Lie algebra of local BRST cohomology classes equipped with the odd Batalin-Vilkovisky antibracket, which has attracted considerable interest recently ; (ii) the Lie algebra of local conserved currents equipped with the Dickey bracket ; and (iii) the Lie algebra of conserved, integrated charges equipped with the Poisson bracket. We show in this paper that the subalgebra of (i) in ghost number -1 and the other two algebras are isomorphic for a field theory without gauge invariance. We also prove that, in the presence of a gauge freedom, (ii) is still isomorphic to the subalgebra of (i) in ghost number -1, while (iii) is isomorphic to the quotient of (ii) by the ideal of currents without charge. In ghost number different from -1, a more detailed analysis of the local BRST cohomology classes in the Hamiltonian formalism allows one to prove an isomorphism theorem between the antibracket and the extended Poisson bracket of Batalin, Fradkin and Vilkovisky.

1 Introduction

The first appearance of an antibracket in the context of Lagrangian field theories can be traced back to the study of the renormalization of Yang-Mills theories when the Ward identities are expressed in terms of the generating functional for one particle irreducible proper vertices [1]. This antibracket has been developped and generalized in the work of Batalin and Vilkovisky [2] on Lagrangian quantization methods for generic gauge theories. The Batalin-Vilkovisky formalism and the antibracket play for instance a fundamental role in the covariant formulation of string field theory [3]. It is therefore of interest to gain a better understanding of the physical significance of this antibracket.

We relate in this paper the Batalin-Vilkovisky antibracket at ghost number minus one both to the bracket introduced by Dickey [4] in the space of local currents, and to the Poisson bracket of conserved charges. More generally, we relate the Batalin-Vilkovisky antibracket for arbitrary values of the ghost number to the extended Poisson bracket appearing in the Hamiltonian formulation of the BRST theory [5, 6].

The paper is organized as follows. In the next section, we review the Batalin-Vilkovisky construction and show that the Batalin-Vilkovisky antibracket naturally induces a well defined odd Lie bracket $\{\cdot,\cdot\}$ in the cohomology classes $H^{*,n}(s|d)$ of the BRST differential s modulo the exterior spacetime differential d in form degree n. The algebra $(H^{*,n}(s|d),\{\cdot,\cdot\})$ possesses a subalgebra $\mathcal{S}$, namely $(H^{-1,n}(s|d),\{\cdot,\cdot\})$.

We then define the Dickey algebra of conserved currents j^μ (section 3) and show that it possesses an ideal, namely the ideal I of non trivial conserved currents for which the charge $Q = \int d^{n-1}x j^0$ is zero on-shell. Such currents are trivial (i.e., on-shell equal to identically conserved currents) when there is no gauge freedom, so that I is effectively zero in that case. They may however be non trivial otherwise. We introduce furthermore the Lie algebra of integrated conserved charges equipped with the covariant Poisson bracket induced by the Dickey bracket.

Isomorphism theorems between $\mathcal{S}$ and the other two Lie algebras in the case of non degenerate field theories are proved in section 4. The modification of these theorems for gauge theories are discussed in section 5. More precisely, we show that $\mathcal{S}$ is still isomorphic to the Dickey algebra, but this algebra itself is now isomorphic to the Lie algebra of conserved charges only after

1

taking the quotient by the ideal I.

In section 6, we investigate the antibracket map for arbitrary ghost number. In order to do so, we go to the extended Hamiltonian formalism and use the fact that the local BRST cohomology group and the associated antibracket map are invariant under this change of description of the theory. The advantage of the Hamiltonian formulation is that the equations of motion are in normal form, which allows one to control the antifield dependence of the local BRST cohomology classes. We show that it is always possible to choose representatives which are at most linear in the antifields of the Hamiltonian description. This allows one to get the general relationship between the antibracket map and the extended Poisson bracket map of the Hamiltonian BRST formalism.

By applying these results to the case of ghost number -1, we find in particular that I is an abelian subalgebra and corresponds to a subspace of the characteristic cohomology associated with the Hamiltonian constraint surface.

2 The antibracket map induced in local BRST cohomology

In the Batalin-Vilkovisky formalism for gauge theories, which we consider for notational simplicity to be irreducible, one introduces, besides the original fields ϕ^i of ghost number 0 and the ghosts C^α of ghost number 1 related to the gauge invariance, the corresponding antifields ϕ_i^* and C_α^* of opposite Grassmann parity and ghost number -1 and -2 respectively [2, 6]. It is natural to define an antibracket by declaring that the fields $\phi^A \equiv (\phi^i, C^\alpha)$ and antifields ϕ_A^* are conjugate :

$$(\phi^A(x), \phi_B^*(y)) = \delta_B^A \delta^n(x - y) \tag{2.1}$$

The antibracket is then given for arbitrary functionals A_1 and A_2 by

$$(A_1, A_2) = \int d^n x \left(\frac{\delta^R A_1}{\delta \phi^A(x)} \frac{\delta^L A_2}{\delta \phi_A^*(x)} - \frac{\delta^R A_1}{\delta \phi_A^*(x)} \frac{\delta^L A_2}{\delta \phi^A(x)} \right). \tag{2.2}$$

The central goal of the formalism is the construction of a proper solution to the master equation

$$(S, S) = 0. \tag{2.3}$$

The functional S is required to start like the classical action S_0, to which one couples through the antifields the gauge transformations with the gauge parameters replaced by the ghosts :

$$S = \int d^n x \hat{\mathcal{L}} = \int d^n x \hat{\mathcal{L}}_0 + \phi_i^* R_\alpha^i C^\alpha + \dots . \tag{2.4}$$

The BRST symmetry is canonically generated in the antibracket through the equation :

$$s = (S, \cdot). \tag{2.5}$$

In order to analyze the properties of the antibracket, it is necessary to have a more precise definition of the functionals to which it applies. We will consider in the following only local functionals. A local functional

$$A[z^a(x)] = \int_X d^n x \ \hat{a}[z^a], \ z^a(x) \to 0 \ for \ x \to \partial X \tag{2.6}$$

is defined as the integral over an orientable domain X of spacetime M^n of a local function $\hat{a}[z^a]$, i.e., a function[1] of x^μ, the fields and antifields $z^a \equiv (\phi^A, \phi_A^*)$ and their derivatives up to some finite order, evaluated for field and antifield histories $z^a(x)$ which appropriately vanish at the boundary ∂X. Note that X can be all of Minkowski space M^n, and that a local function corresponds to a function on the finite dimensional "jet-space" $M^n \times V^k$ with coordinates $x^\mu, \partial_{(\nu)} z^a, |\nu| \leq k$ (see appendix A for more details). The space of local functionals so defined can be proved (see for instance [8, 6]) to be isomorphic to the space of equivalence classes of local functions $\hat{a}$ modulo total divergences $\partial_\mu j^\mu$, for some arbitrary local current j^μ. The total derivative ∂_μ is defined in multiindex notation by

$$\partial_\mu = \frac{\partial^L}{\partial x^\mu} + \partial_{\mu(\nu)} z^a \frac{\partial^L}{\partial(\partial_{(\nu)} z^a)}. \tag{2.7}$$

One can furthermore prove that a local function is a total divergence if and only if its Euler-Lagrange derivatives vanish (see e.g. [8]).

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[1] We will not be too precise about the nature of the field dependence of the local functions (polynomiality or smooth dependence). Similarily, we will not specify whether one should consider polynomials or infinite formal series in the antifields and their derivatives [7], since most aspects we will consider are really independent of these considerations. For simplicity, we will assume however that all the fields live on a star-shaped space.

Turning to form notations, $\hat{a} \to a = d^n x \hat{a}$ and introducing the spacetime exterior derivative $d = dx^\mu \partial_\mu$, the space of local functions can be identified with the cohomology group $H^n(d)$ of the differential d in form degree n in the space of local, form valued functions.

It is easy to verify that the antibracket of two local functionals is also a local functional. Thus the antibracket induces a well defined map in the cohomology group $H^n(d)$,

$$\{\cdot,\cdot\} : H^n(d) \times H^n(d) \longrightarrow H^n(d) \tag{2.8}$$

This bilinear map enherits from the antibracket the property of being a true, odd, Lie bracket. If we denote by $[a]$ the cohomological class of the n-form a in $H^n(d)$, one may view the antibracket in $H^n(d)$ as arising from a local antibracket in the space of local functions defined as follows,

$$\{\hat{a}_1, \hat{a}_2\} = \frac{\delta^R \hat{a}_1}{\delta \phi^A} \frac{\delta^L \hat{a}_2}{\delta \phi_A^*} - \frac{\delta^R \hat{a}_1}{\delta \phi_A^*} \frac{\delta^L \hat{a}_2}{\delta \phi^A} \tag{2.9}$$

$$\{[a_1], [a_2]\} = [d^n x \{\hat{a}_1, \hat{a}_2\}]. \tag{2.10}$$

In (2.9), $\delta/\delta\phi^A$ is the Euler-Lagrange derivative defined by

$$\frac{\delta}{\delta \phi^A} = (-\partial)_{(\nu)} \frac{\partial}{\partial(\partial_{(\nu)} \phi^A)}, \tag{2.11}$$

with $(-\partial)_{(\nu)} = (-)^{|\nu|} \partial_{(\nu)}$. While the bracket (2.8) in $H^n(d)$ is a true bracket, the local antibracket (2.9) in the space of local functions is graded symmetric, but satisfies the graded Leibnitz rule and Jacobi identity only up to total divergences (see Appendix B).

It is clear that the antibracket for the integrands that gives rise to the antibracket in $H^n(d)$ is not unique, but expressions differing from the one in (2.9) by a total divergence are also admissible. This is the case for instance for the following expression (see appendix B),

$$\{\hat{a}_1, \hat{a}_2\}_{alt} = \partial_{(\nu)}\left(\frac{\delta^R \hat{a}_1}{\delta \phi^A}\right) \frac{\partial^L \hat{a}_2}{\partial(\partial_{(\nu)} \phi_A^*)} - \partial_{(\nu)}\left(\frac{\delta^R \hat{a}_1}{\delta \phi_A^*}\right) \frac{\partial^L \hat{a}_2}{\partial(\partial_{(\nu)} \phi^A)}, \tag{2.12}$$

which satifies a graded Leibnitz rule in the second argument, but is only graded symmetric up to a total divergence. [There is no expression for the

4

local antibracket in the space of local functions that satisfies strictly all the properties of an ordinary odd Lie bracket, without extra divergences.]

In the Batalin-Vilkovisky formalism, one introduces additional fields, the ghosts and antifields. Quantities of direct physical interest are recovered by considering the cohomology classes of the BRST differential s. The identification of local functionals with the cohomology group $H^n(d)$ implies that the BRST cohomology for local functionals is given by $H^*(s, H^n(d))$. This last group is isomorphic to the relative cohomology group $H^{*,n}(s|d)$ of s modulo d in form degree n evaluated in the space of form valued local functions. Due to the fact that the BRST symmetry acting on a local function is canonically generated through the formula

$$s\hat{a} = \{\hat{\mathcal{L}}, \hat{a}\}_{alt}, \tag{2.13}$$

it is straightforward[2] to verify that the local antibracket induces a well defined odd Lie bracket in the relative cohomology group of s modulo d :

$$\{\cdot, \cdot\} : H^{n,g_1}(s|d) \times H^{n,g_2}(s|d) \longrightarrow H^{n,g_1+g_2+1}(s|d)$$
$$\{[a_1], [a_2]\} = [d^n x \{\hat{a}_1, \hat{a}_2\}]. \tag{2.14}$$

An inspection of the various possible cases shows that it is only for ghost number -1 that this map associates to two cohomology classes a cohomology class of the same type, i.e., of same ghost number. The subspace $H^{-1,n}(s|d)$ equipped with the antibracket defines a subalgebra of $H^{n,*}(s|d)$ which we denote by $\mathcal{S}$,

$$\mathcal{S} = (H^{-1,n}(s|d), \{\cdot, \cdot\}). \tag{2.15}$$

3 The Dickey bracket

Let Σ_k be the stationary surface, i.e., the surface defined by the equations

$$\partial_{(\lambda)}(\delta \hat{\mathcal{L}}_0 / \delta \phi^i) = 0, \tag{3.1}$$

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[2]One uses the facts that (i) $\{\cdot, \cdot\}_{alt}$ differs from $\{\cdot, \cdot\}$ by a total divergence, (ii) that $\{\cdot, \cdot\}$ satisfies the graded Jacobi up to a total divergence, and (iii) that Euler-Lagrange derivatives annihilate total divergences.

(with $|\lambda| \leq k - 2$ for second order equations) in the spaces $M \times F^k$ with coordinates $x^\mu, \partial_{(\mu)}\phi^i, |\mu| \leq k$.

The vector space of (equivalence classes of) inequivalent Lagrangian conservation laws is defined by

$$\{j^\mu, \partial_\mu j^\mu \approx 0, \text{modulo the identification } j^\mu \sim |_\Sigma j^\mu + \partial_\nu S^{[\nu\mu]}\}, \qquad (3.2)$$

where the j^μ are local functions. In form notations, we get equivalence classes $[j]$ of $n-1$ forms whose pull-back to the stationary surface is d-closed, where two such forms have to be identified if they differ by the exterior derivative of an $n-2$ form on the stationary surface :

$$[j] \in H^{n-1}(d^*, \Omega(\Sigma)). \qquad (3.3)$$

Inequivalent conserved currents belong, by definition, to the so called characteristic cohomology of the stationary surface in form degree $n - 1$.

The standard regularity conditions are that locally in the jet-space, the equations $\delta\hat{\mathcal{L}}_0/\delta\phi^i$ and their derivatives can be split into two groups, the "independent equations" which can be taken locally as a new coordinate system on the jet-space replacing some of the fields and their derivatives, and the "dependent equations" which hold as a consequence of the independent ones. One then can prove [8, 6] that a function which vanishes on the stationary surface can be written as a linear combination of the equations defining the surface, hence

$$\partial_\mu j^\mu = X^{i(\lambda)}\partial_{(\lambda)}\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^i} \qquad (3.4)$$

for some local functions $X^{i(\lambda)}$. This equation does not determine $X^{i(\lambda)}$ completely, one is for instance free to add functions of the form

$$Y^{i(\lambda)j(\nu)}\partial_{(\nu)}\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^j} \qquad (3.5)$$

with Y antisymmetric under the exchange of the pairs $i(\lambda)$ and $j(\nu)$[3].

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[3]This exhausts the arbitrariness of the functions $X^{i(\lambda)}$ only in the case where the equations and their derivatives are independent [8, 6] ; in the general case, one has to take care also of the Noether identities, as shown below.

The characteristic [8, 4] of the equivalence class of conservation laws described by $[j]$ is defined by the equivalence class of local functions of the form $X^i = (-\partial)_{(\lambda)} X^{i(\lambda)}$, where two sets X^i's of local functions have to be identified if they differ by a function of the form

$$(-\partial_{(\lambda)})[Y^{i(\lambda)j(\nu)} \partial_{(\nu)} \frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^j}]. \tag{3.6}$$

It is straightforward to verify that the characteristic does not depend on the choice of the representative for j^μ. Let δ_X be the evolutionary vector field defined by X^i :

$$\delta_X = \partial_{(\lambda)} X^i \frac{\partial}{\partial(\partial_{(\lambda)} \phi^i)}. \tag{3.7}$$

Note that X^i and δ_X satisfy the equations

$$X^i \frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i} = \partial_\mu j'^\mu, \ \delta_X \hat{\mathcal{L}}_0 = \partial_\mu j''^\mu, \tag{3.8}$$

with j'^μ, j''^μ in the same equivalence class as j. This means that the characteristics define variational symmetries, i.e., symmetries of the action. In the non degenerate case, one can then prove directly that there is a one to one correspondence between inequivalent symmetries of the action and inequivalent conservation laws (Noether's theorem) [8], but we will not do so here because it is also a direct consequence of our analysis in the next section.

The Dickey bracket (3.2) in the space of inequivalent conservation laws is defined by [4]

$$\{[j_1], [j_2]\}_D = -[\delta_{X_1} j_2]. \tag{3.9}$$

By using properties of the Euler-Lagrange derivatives, one finds the following equivalent expressions (see [4] and Appendix B):

$$\{[j_1], [j_2]\}_D = [\delta_{X_2} j_1] = \frac{1}{2}[\delta_{X_2} j_1 - \delta_{X_1} j_2]$$

$$= [-\frac{(\tilde{\nu})_\mu + 1}{|\nu| + 1} \partial_{(\nu)} [\delta_{X_2} (\frac{\delta \mathcal{L}_0}{\delta(\partial_{(\nu)\mu} \phi^i)}) X_1^i - \delta_{X_1} (\frac{\delta \mathcal{L}_0}{\delta(\partial_{(\nu)\mu} \phi^i)}) X_2^i]$$

$$\frac{1}{(n-1)!} \varepsilon_{\mu_1 \ldots \mu_n} dx^{\mu_2} \ldots dx^{\mu_n}], \tag{3.10}$$

7

where $(\nu)_\mu$ denotes the number of occurences of μ in the multiindex (ν). This last expression corresponds to the contraction of the horizontal $(n-1)-$ and vertical $2-$ form

$$\Omega = \frac{1}{(n-1)!}\varepsilon_{\mu_1\ldots\mu_n}dx^{\mu_2}\ldots dx^{\mu_n}\frac{\tilde{\mu}+1}{|\nu|+1}\partial_{(\nu)}[d_V(\frac{\delta\mathcal{L}_0}{\delta(\partial_{(\nu)\mu_1}\phi^i)})d_V\phi^i] \quad (3.11)$$

with the evolutionary vector fields δ_{X_1} and δ_{X_2}. This formula involves the vertical derivatives and the higher order Euler operators defined for instance in [8, 4] (see also appendix A and B).

Again, in the non degenerate case, one can prove directly that the Dickey bracket is a well defined Lie bracket in the space of inequivalent conserved currents (see [4]) ; namely, it is unambiguous in the quotient space, antisymmetric and satisfies the Jacobi identity. Alternatively, these properties follow from the isomorphism theorem proved in the next section.

Among the conserved currents, one may distinguish between those for which j^0 is trivial, i.e., of the form $j^0 \approx \partial_m S^{m0}$. The corresponding Noether charge $Q = \int d^{n-1}x j^0$ is zero on the stationary surface. These currents form an ideal for the Dickey bracket since $\delta_X j^0$ is trivial if j^0 is trivial. We call this ideal the ideal of "conserved currents without charge" and we denote it by I.

As we shall show in the next section, the ideal I is trivial in the absence of gauge symmetry. That is, if a conserved current has a vanishing Noether charge, then, it is trivial, i.e., on-shell equal to an identically conserved current. But this may not be so in the presence of gauge freedom, for which there exist non trivial currents in I.

The third algebra that we shall introduce is the algebra of conserved, integrated charges, $Q = \int d^{n-1}x j^0$, $\partial_0 j^0 = -\partial_k j^k$ for some spatial current j^k, with the identification of two such charges if they agree on the stationary surface. By using the Hamiltonian formalism, one may equip this algebra with a well defined even bracket, namely, the standard Hamiltonian Poisson bracket. We denote this algebra by $\mathcal{Q}$. It is clear that $\mathcal{Q}$ is isomorphic as a vector space to the quotient of the space of conserved currents by the ideal I. We shall prove furthermore that the Poisson bracket is just the corresponding induced Dickey bracket.

4 Isomorphisms in the case of non-degenerate Lagrangian field theory

In the absence of gauge invariance, the only additional fields in the Batalin-Vilkovisky construction besides the original ϕ^i, which we assume for simplicity to be bosonic, are the antifields ϕ_i^*. The original action $S[\phi^i] = \int d^n x \, \hat{\mathcal{L}}_0[\phi^i]$ is by itself a proper solution of the master equation generating the BRST symmetry

$$s\phi_i^* = \frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i}, \;\; s\phi^i = 0, \;\; s\partial_\mu = \partial_\mu s, \tag{4.1}$$

which reduces to the so called Koszul-Tate differential δ [6]. In the non degenerate case, the equations of motion and their derivatives can be taken locally as first coordinates in a new coordinate system replacing some of the fields and their derivatives. One can prove [6] that the BRST cohomology in the spaces $C^\infty(\mathbf{R}^n \times F^k) \times \mathbf{R}[\partial_{(\nu)}\phi_i^*]$ (with $|\nu| \leq k - 2$ for second order equations) is given by smooth functions defined on the stationary surface[4] : $H^0(\delta) \simeq C^\infty(\mathbf{R}^n \times \Sigma^k)$ and $H^g(\delta) = 0, g \neq 0$.

In the new coordinate system, where the equations and their derivatives are taken as new coordinates, we denote by $I_0 = \{x_a\}$ the set of fields and their derivatives needed to complete the coordinate system. Let us assume that the non degenerate theory is of Cauchy order 1, meaning that $\partial_k x_a \in I_0$ for $k \geq 1$. One can then prove [7] that, apart from $H^{0,n}(s|d)$, which corresponds to local functionals defined on the stationary surface, the only non trivial local BRST cohomology classes are in ghost number -1 and form degree n.

By integrations by parts, the representatives of $H^{-1,n}(s|d)$ can be assumed to be of the characteristic form

$$a = d^n x \, \phi_i^* X^i[\phi^i] \tag{4.2}$$

for local functions X^i. The cocycle condition reads

$$\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i} X^i = \partial_\mu j^\mu, \tag{4.3}$$

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[4]One says that the Koszul-Tate differential δ provides a homological resolution of the functions defined on the stationary surface (see also Appendix A).

and implies that the field variation $\delta\phi^i = X^i$ defines a variational symmetry. Furthermore, to a trivial representative of $H^{-1,n}(s|d)$ corresponds a variational symmetry which is given by an "antisymmetric" combination of the equations of motions as in (3.6)[5]. The space $H^{-1,n}(s|d)$ is accordingly given by inequivalent variational symmetries or characteristics of inequivalent conservation laws[6].

The local antibracket map for such representatives of $H^{-1,n}(s|d)$ is given by :

$$\{[d^n x\ \phi_i^* X_1^i], [d^n x\ \phi_i^* X_2^i]\} = [d^n x\ \phi_i^* [X_1, X_2]_L^i]$$

$$[X_1, X_2]_L^i = \frac{\partial X_1^i}{\partial\phi_{(\mu)}^j}\partial_{(\mu)}(X_2^j) - \frac{\partial X_2^i}{\partial\phi_{(\mu)}^j}\partial_{(\mu)}(X_1^j) = \delta_{X_2}X_1^i - \delta_{X_1}X_2^i. \qquad (4.4)$$

Hence we find that, in ghost number -1, the local antibracket map corresponds to the traditional, even Lie bracket for inequivalent variational symmetries under characteristic form given in [8]. Since the Lie bracket for evolutionary vector fields is induced by the commutator for vector fields, we get :

Theorem 4.1 *The odd Lie algebra $\mathcal{S} = (H^{-1,n}(s|d), \{\cdot, \cdot\})$ is isomorphic to the algebra of inequivalent variational symmetries equipped with the bracket induced by the commutator for vector fields.*

Using the acyclicity of $s = \delta$ [6] at negative ghost numbers and the triviality of the cohomology of d in form degree $p < n$ ($H^p(d) = \delta_0^p \mathbf{R}$, see e.g. [8]), we can easily prove the isomorphism

$$H^{-1,n}(\delta|d) \simeq H^{n-1,0}(d|\delta)/\delta_1^n \mathbf{R}. \qquad (4.5)$$

This follows from a general relationship for relative cohomology groups proved in [10]. The last space corresponds to the space of inequivalent conserved currents. Indeed, the cocycle condition implies that representatives must be

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[5]A trivial variational symmetry vanishes on the stationary surface. Under certain assumptions [7], one can prove that vice-versa every variational symmetry which vanishes on the stationary surface corresponds to a trivial representative of $H^{-1,n}(s|d)$, i.e., an "antisymmetric" combination of the equations of motion.

[6]This is the formalisation in the appropriate jet space of the idea that functions linear in the antifields define tangent vectors [9], the physically relevant ones here being those that are "tangent" to the stationary surface.

$n-1$-forms which restrict to closed forms on the stationary surface, while the coboundary condition requires two such currents to be considered as equivalent if they differ on this surface by the exterior derivative of a $n-2$ form, i.e., the divergence of a "superpotential" in dual notation, or by a constant in 1 dimension.

The above isomorphism is explicitly given by associating to a representative a of the first space the representative j of the second space in the equation $sa+dj = 0$. Furthermore, the antibracket map induces through this isomorphism a well defined Lie bracket in the space of inequivalent conserved currents. An explicit calculation (Appendix B) shows that the corresponding bracket is just given by the Dickey bracket. Hence,

Theorem 4.2 *The odd Lie algebra $\mathcal{S}$ is isomorphic to the space of inequivalent conservation laws equipped with the Dickey bracket.*

There is no contradiction in the fact that the isomorphism relates an odd bracket to an even bracket, because there is at the same time a shift in the degree (from odd (-1) to even (0)).

Combining theorems 4.1 and 4.2, we get the full Noether theorem :

Corollary 4.1 *There is a Lie algebra isomorphism between inequivalent conservation laws and inequivalent variational symmetries.*

The ideal I of currents of the second set is trivial. Indeed, the coboundary condition allows us to take all the j^k to depend on the x_a alone. Because $\partial_k x_a$ depends also on x_a and not on equations of motion, one must have $\partial_k j^k = 0$ identically, which implies that $j^k = \delta_2^n \mathbf{R} + \partial_m S^{[mk]}$. Hence, the Dickey algebra and the space of inequivalent, integrated conserved charges are isomorphic as vector spaces. That the induced Dickey bracket in the space $\mathcal{Q}$ corresponds to the Poisson bracket in $\mathcal{Q}$ in the Hamiltonian formalism is a consequence of the analysis in section 6. Alternatively, it could be proved directly along the lines of [11], by taking furthermore locality into account. Hence,

Theorem 4.3 *In dimensions different from 2, if the theory is of Cauchy order 1, the Dickey algebra of inequivalent conserved currents is isomorphic to the algebra of inequivalent conserved charges equipped with the Poisson bracket.*

5 Gauge theories. Ghost number -1

The advantage of the cohomological reformulation of Noether's theorem in equation (4.5) is that one can extend this theorem in a straightforward way to gauge theories, which are not covered by the analysis in [8, 4]. One can prove that the subalgebra $\mathcal{S}$ is isomorphic to the algebra $\mathcal{R} = (H_1^n(\delta|d), \{\cdot, \cdot\}_R)$, where the cohomology group $H_1^n(\delta|d)$ involves only the original fields and the antifields, but no ghosts, δ being the Koszul-Tate part of s and the degree of δ the antighost number, which is minus the ghost number (for a function that does not involve the ghosts). The restricted antibracket map $\{\cdot, \cdot\}_R$ is the antibracket map restricted to the original fields ϕ^i and the antifields ϕ_i^*.

The differential δ acts non trivially on the antifields of higher order. In the case of irreducible gauge theories, its action on C_a^* is given by

$$\delta\partial_{(\lambda)}C_a^* = \partial_{(\lambda)}[R_a^{+i(\nu)}\partial_{(\nu)}\phi_i^*], \tag{5.1}$$

where the operators $R_a^{+i(\nu)}\partial_{(\nu)}$ define the Noether identities of the theory, i.e.,

$$R_a^{+i(\nu)}\partial_{(\nu)}\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^i} = 0. \tag{5.2}$$

This additional piece maintains $\delta^2 = 0$ and guarantees that δ still defines a homological resolution of the functions defined on the constraint surface, implying for instance that equation (4.5) still holds.

If we still want theorem 4.1 to hold, the definition of δ requires that we change the notion of a trivial variational symmetry ; they have to correspond to X^i's which are "antisymmetric" combinations of the equations of motion up to a gauge transformation where the gauge parameters are replaced by arbitrary local functions :

$$X^i = (-\partial_{(\lambda)})[Y^{i(\lambda)j(\nu)}\partial_{(\nu)}\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^j}] + R_a^{i(\nu)}\partial_{(\nu)}f^a. \tag{5.3}$$

The operators $R_a^{i(\nu)}\partial_{(\nu)}$ are the adjoints of the operators defining the Noether identities and define the gauge transformations.

With this modification of the space of inequivalent variational symmetries, theorems 4.1, 4.2 and corollary 4.1 hold as in the case with no gauge invariance.

The ideal I however is not trivial in the case of gauge theories, because the theory is no longer of Cauchy order 1. For instance, the current $j^\mu = F^{0\mu} = (0, F^{0k})$ in free Maxwell's theory belongs to I since $\int j^0 d^{n-1}x = 0$ but $F^{0k} \neq \partial_m S^{[km]}$ (even weakly). Theorem 4.3 becomes :

Theorem 5.1 *The Dickey algebra of conserved currents modulo the ideal I is isomorphic to the algebra of inequivalent conserved charges equipped with the Poisson bracket.*

The proof that the induced Dickey bracket is in fact the ordinary Poisson bracket in the Hamiltonian formalism again follows from the reasoning given in the next section.

6   Gauge theories. General analysis

The previous theorems relate the antibracket and the Poisson bracket at particular values of the ghost number. In order to fully prove them, we shall first put them in a more general setting. Indeed, these theorems can be extended to arbitrary values of the ghost number.

To relate the antibracket and the Poisson bracket for all values of the ghost number, one first uses the invariance of the local BRST cohomology group with respect to the introduction of auxiliary fields and generalized auxiliary fields as shown in [7]. One proves by an analoguous reasoning that the same is true for the antibracket map induced in cohomology. This implies that one can go to the total Hamiltonian formalism and then to the extended Hamiltonian formalism, which we will assume to be local [6], and describe the solution of the master equation in terms of the Batalin-Fradkin-Vilkovisky framework.

Let us recall that in this framework, a central object is the extended Poisson bracket $[\cdot, \cdot]_P$ for which the ghosts C^a and the ghost momenta $\mathcal{P}_b$ are considered as conjugate dynamical variables in addition to the usual fields and their momenta. One then constructs out of the constraints, which we assume for simplicity to be irreducible and first class, the BRST charge $\Omega = \int d^{n-1}x\ \omega$ which is a local functional in space verifying $[\Omega, \Omega]_P = 0$. The Hamiltonian $H = \int d^{n-1}x\ h$ verifying $[\Omega, H]_P = 0$ is also a local functional in space and these two functionals depend only on the fields $\tilde{\phi}^A \equiv \phi^i, \pi_j, C^a, \mathcal{P}_b$ and their spatial derivatives.

The functionals in space are replaced by spatial functions in the same way as in the spacetime case, which leads to a local extended Poisson bracket $\{\cdot,\cdot\}_P$ defined through spatial Euler Lagrange derivatives. The BRST charge Ω generates the symmetry $s_\omega = \{\omega,\cdot\}_{P,alt}$ where $\{\cdot,\cdot\}_{P,alt}$ is defined in a way analoguous to $\{\cdot,\cdot\}_{alt}$ in (2.12). The local extended Poisson bracket induces a well defined even Lie bracket, the Poisson bracket map, in the cohomology group of s_ω modulo the spatial exterior derivative $\tilde{d}$.

The symmetry s_ω is only a part of the BRST symmetry which is isomorphic to the BRST symmetry of the initial Lagrangian system through the elimination of (generalized) auxiliary fields. The complete BRST symmetry is generated through the solution of the master equation in the extended Hamiltonian formalism given by [12, 6]

$$S_H[\tilde{\phi}^A, \tilde{\phi}^*_A] = \int dt d^{n-1}x (-\frac{1}{2}\dot{\tilde{\phi}}^A(\sigma^{-1})_{AB}\dot{\tilde{\phi}}^B - h - \{\tilde{\phi}^*_A\tilde{\phi}^A, \omega\}_{P,alt}), \quad (6.1)$$

where we have introduced the notation

$$\sigma^{AB} = \begin{pmatrix} 0 & 0 & \delta_j{}^i & 0 \\ 0 & 0 & 0 & -\delta_b{}^a \\ -\delta_j{}^i & 0 & 0 & 0 \\ 0 & -\delta_b{}^a & 0 & 0 \end{pmatrix}. \quad (6.2)$$

Explicitly, the BRST symmetry $s_H = \{S_H, \cdot\}_{alt}$ reads

$$s_H = \partial_{(\mu)}(\frac{\tilde{\delta}^R\omega}{\delta\tilde{\phi}^A})\sigma^{AB}\frac{\partial^L}{\partial(\partial_{(\mu)}\tilde{\phi}^B)} + \partial_{(\mu)}\mathcal{L}_A\frac{\partial^L}{\partial(\partial_{(\mu)}\tilde{\phi}^*_A)}, \quad (6.3)$$

where the tilded Euler-Lagrange derivatives are restricted to spatial derivatives only and

$$\mathcal{L}_A \equiv -\dot{\tilde{\phi}}^B(\sigma^{-1})_{BA} - \frac{\tilde{\delta}^R h}{\delta\tilde{\phi}^A} - \frac{\tilde{\delta}^R}{\delta\tilde{\phi}^A}(\{\tilde{\phi}^*_B\tilde{\phi}^B, \omega\}_{P,alt})). \quad (6.4)$$

Note that in the proper solution S_H to the master equation in the extended Hamiltonian formalism, we have made the identification of minus the antifield $-\lambda^*_a$ of the Lagrange multiplier for the first class constraints with the ghost momenta $\mathcal{P}_a$. This implies that in terms of the new antifields, the Koszul-Tate part is now associated to the surface $\mathcal{L}_A(\tilde{\phi}^* = 0) = 0$ and not with

14

the gauge invariant, original, Hamiltonian equations of motion. The part in resolution degree[7] 0 with respect to the new antifields is given by

$$\gamma = s_\omega^0 - \partial_{(\mu)} \frac{\tilde{\delta}^R}{\delta \tilde{\phi}^A}(\{\tilde{\phi}_B^* \tilde{\phi}^B, \omega\}_{P,alt})) \tag{6.5}$$

and the BRST differential has no contributions in higher resolution degree, contrary to what may happen in the old resolution degree. Here, s_ω^0 is defined by the first term on the right hand side of equation (6.3) and coincides with s_ω when acting on a function involving no time derivatives of the fields. Evaluating the action of $s_\omega^{(0)}$ on ϕ^i, π_j and putting to zero the ghost momenta $\mathcal{P}_a$ reproduces the gauge transformations of these fields with gauge parameters replaced by the ghosts C^a.

One then investigates the local BRST cohomology groups $H(s_H|d)$. A first step is the following theorem.

Theorem 6.1 *The ordinary BRST cohomology depending on the fields ϕ^A, the antifields ϕ_A^* and their derivatives is isomorphic to the cohomology of s_ω depending on the fields $\tilde{\phi}^A$ and their spatial derivatives :*

$$H(s, [\phi^A, \phi_A^*]) \simeq H(s_H, [\tilde{\phi}^A, \tilde{\phi}_A^*]) \simeq H(s_\omega, [\tilde{\phi^A}]). \tag{6.6}$$

In other words, in a s_H cocycle, one can get rid of the temporal derivatives and of the antifields through the addition of a s_H coboundary. For a proof of this theorem, see Appendix C.

Starting from the bottom of the descent equations, one then proves (see again Appendix C) that a non trivial cocycle modulo d, a, $s_H a + db = 0$, given by $a = \tilde{a} + dt a^0$, where $\tilde{a}$ does not involve the differential dt, can be characterized by

$$a = dt(-\{\phi_A^* \phi^A, \tilde{b}_0\}_{P,alt} + a_0^0) + \tilde{a}_0, \tag{6.7}$$

verifying

$$s_\omega \tilde{a}_0 + \tilde{d\tilde{b}_0} = 0 \tag{6.8}$$

$$s_\omega a_0^0 + \tilde{d} b_0^0 - \frac{\partial}{\partial t} \tilde{b}_0 + \{h, \tilde{b}_0\}_{P,alt} = 0. \tag{6.9}$$

---

[7]The resolution degree is the degree associated to the Koszul-Tate differential [6].

Here, $\tilde{a}_0, \tilde{b}_0, a_0^0$ and b_0^0 contain no antifields and no time derivatives of the fields, while $\tilde{b}_0$ and b_0^0 satisfy analoguous equations to $\tilde{a}_0$ and a_0^0 for some $\tilde{m}_0, m_0^0$. In maximum form degree n, there is of course no $\tilde{a}$ and at the bottom, say n, of the descent equations, $\tilde{n}_0$ and n_0^0 are s_ω-cocycles[8].

In the coboundary condition for such cocycles $a = s_H c + de$, we have

$$c = dt(-\{\tilde{\phi}_A^*\tilde{\phi}^A, \tilde{e}_0\}_{P,alt} + e_0^0) + \tilde{c}_0, \qquad (6.10)$$

giving the conditions

$$\tilde{a}_0 = s_\omega \tilde{c}_0 + \tilde{d}\tilde{e}_0 \qquad (6.11)$$

$$a_0^0 = -s_\omega c_0^0 - \tilde{d}e_0^0 + \frac{\partial}{\partial t}\tilde{e}_0 - \{h, \tilde{e}_0\}_{P,alt} \qquad (6.12)$$

where $\tilde{c}_0, \tilde{e}_0, c_0^0$ and e_0^0 again contain no antifields and no time derivatives of the fields, with analoguous equations holding for $\tilde{b}_0, b_0^0$ in terms of $\tilde{e}_0, e_0^0, \tilde{f}_0, f_0^0$. In maximum form degree, there is no $\tilde{a}, \tilde{c}$ and equation (6.11) is trivially satisfied.

In order to characterize the local BRST cohomology groups $H^{g,k}(s_H|d)$, one can first find a basis for the vector space $H^{g,k}(s_\omega|\tilde{d})$ in the space of antifield and time derivative independent local forms with only spatial differentials (most general non trivial solution for $\tilde{a}_0$). One then finds a basis for $H^{g+1,k-1}(s_\omega|\tilde{d})$ (most general non trivial solution for $\tilde{b}_0$). One finally considers the subspace $l[H^{g+1,k-1}(s_\omega|\tilde{d})]$ for which equation (6.9) admits a particular solution a_{0P}^0. The general non trivial form for a_0^0 is then given by $a_0^0 = a_{0P}^0 + \bar{a}_0^0$ where $\bar{a}_0^0$ belongs to $r[H^{g,k-1}(s_\omega|\tilde{d})]$, which is the subspace of $H^{g,k-1}(s_\omega|\tilde{d})$ remaining non trivial under the more general coboundary condition (6.12).

We thus get the following result on the relationship between the local BRST cohomology groups in Lagrangian and Hamiltonian formalism :

Theorem 6.2 *The local BRST cohomology groups are isomorphic to the direct sum of the following three local cohomology groups of the Hamiltonian formalism :*

$$H^{g,k}(s|d) \simeq H^{g,k}(s_H|d) \simeq$$
$$H^{g,k}(s_\omega|\tilde{d}) \oplus l[H^{g+1,k-1}(s_\omega|\tilde{d})] \oplus r[H^{g,k-1}(s_\omega|\tilde{d})]. \qquad (6.13)$$

---

[8]These equations have been first used in [13] to compare anomalies in the Hamiltonian and the Lagrangian formalism.

Note that in maximal form degree n, the first group of the last line vanishes. This decomposition is in general quite difficult to achieve in practice since it requires the resolution of complicated equations. However, it corresponds to the natural resolution of the spatio-temporal descent equations in the Hamiltonian formalism and it is useful in principle, since it enables one to relate the bracket and the antibracket.

Remark : The groups with prefix r and l appear also in the covariant analysis of the descent equations in the following way. The descent equations provide a homomorphism $\mathcal{D} : H^{g,k}(s|d) \longrightarrow H^{g+1,k-1}(s|d)$ with $\mathcal{D}[a] = [b]$ for $sa + db = 0 (\longrightarrow sb + dc = 0)$. The kernel of $\mathcal{D}$ can easily be shown to consist of the vector space $H^{g,k}(s)$ seen as a subspace of $H^{g,k}(s|d)$, i.e., the equivalence classes of s-cocycles with equivalence relation determined by s modulo d exactness. We denote this kernel by $r[H^{g,k}(s)]$.

The image of $\mathcal{D}$ is given by the classes $[b] \in H^{g+1,k-1}(s|d)$, which can be lifted, i.e., such that there exists a with $sa + db = 0$. We denote this space by $l[H^{g+1,k-1}(s|d)]$.

This implies the isomorphism

$$H^{g,k}(s|d) \simeq l[H^{g+1,k-1}(s|d)] \oplus r[H^{g,k}(s)] \tag{6.14}$$

and, by iteration,

$$H^{g,k}(s|d) \simeq \oplus_{i=0}^{k} l^{i} r[H^{g+i,k-i}(s)], \tag{6.15}$$

where in the last space $(i = k)$ one can forget the r, because there are no d exact terms in form degree 0. Note that since $H^0(d) = \mathbf{R}$, if $g = -k$, the last space has to be replaced by the space $\{e, se = c, e \sim e + sf + c'; c, c' \in \mathbf{R}\}$ which is isomorphic to $H^0(s)/\mathbf{R}$.

In the Hamiltonian case above, we consider only the part of the descent equations involving the exterior derivative with respect to time : $d^0 = dt \ (d/dt)$.□

We now use theorem 6.2 to derive information on the antibracket from the Poisson bracket induced in $H(s_\omega|\tilde{d})$. On the representatives of the local BRST cohomology groups determined by equations (6.7)-(6.12), the local antibracket gives

$$\{\hat{a}_1, \hat{a}_2\} = \{\phi_A^* \phi^A, \{\hat{\tilde{b}}_1, \hat{\tilde{b}}_2\}_P\}_{P,alt} - \{\hat{a}_1^0, \hat{\tilde{b}}_2\}_P - (-)^{\varepsilon_{\hat{b}_1}} \{\hat{\tilde{b}}_1, \hat{a}_2^0\}_P. \tag{6.16}$$

Hence, (i) the antibracket map can be entirely rewritten in terms of the local Poisson bracket and (ii) it is non trivial only if $l[H^{*,n-1}(s_\omega|\tilde{d})]$ is non trivial.

More precisely, according to the split of $H^{*,n}(s|d)$ in (6.13) to which corresponds the split of a^0 into a_P^0 and $\bar{a}_0$, we see that the antibracket map (2.14) is completely determined by the local Poisson bracket map induced in

$$\{\cdot,\cdot\}_P : l[H^{g_1+1,n-1}(s_\omega|\tilde{d})] \times l[H^{g_2+1,n-1}(s_\omega|\tilde{d})]$$
$$\longrightarrow l[H^{g_1+g_2+2,n-1}(s_\omega|\tilde{d})] \qquad (6.17)$$

and by the local Poisson bracket map in

$$\{\cdot,\cdot\}_P : l[H^{g_1+1,n-1}(s_\omega|\tilde{d})] \times r[H^{g_2,n-1}(s_\omega|\tilde{d})]$$
$$\longrightarrow r[H^{g_1+g_2+1,n-1}(s_\omega|\tilde{d})]. \qquad (6.18)$$

Hence the antibracket map is determined by the following matrix in maximum spatial form degree $n-1$:

$$\begin{pmatrix} \{l[H^{g_1+1}(s_\omega|\tilde{d})], l[H^{g_2+1}(s_\omega|\tilde{d})]\}_P & (-)^{\varepsilon_{g_1}+1}\{l[H^{g_1+1}(s_\omega|\tilde{d})], r[H^{g_2}(s_\omega|\tilde{d})]\}_P \\ \{r[H^{g_1}(s_\omega|\tilde{d})], l[H^{g_2+1}(s_\omega|\tilde{d})]\}_P & 0 \end{pmatrix} (6.19)$$

Equations (6.18) and (6.19) mean in particular that $r[H^{*,n-1}(s_\omega|\tilde{d})]$ is an abelian subalgebra and an ideal in the odd Lie algebra $(H^{*,n}(s|d), \{\cdot,\cdot\})$. We have thus proved :

Theorem 6.3 *The odd Lie algebra $(H^{*,n}(s|d), \{\cdot,\cdot\})$ is isomorphic to the semi-direct sum of the abelian Lie algebra $r[H^{*,n-1}(s_\omega|\tilde{d})]$ and the Lie algebra $(l[H^{*,n-1}(s_\omega|\tilde{d})], \{\cdot,\cdot\}_P)$, where the action of $(l[H^{*,n-1}(s_\omega|\tilde{d})]$ on $r[H^{*,n-1}(s_\omega|\tilde{d})]$ is determined by the Poisson bracket map from one space to the other. By taking the quotient, the following isomorphism is seen to hold :*

$$(H^{*,n}(s|d)/r[H^{*,n-1}(s_\omega|\tilde{d})], \{\cdot,\cdot\}) \simeq (l[H^{*,n-1}(s_\omega|\tilde{d})], \{\cdot,\cdot\}_P). \qquad (6.20)$$

The consequences of this result in the particular case of conserved currents, i.e., for $g_1 = g_2 = -1, k = n$ are as follows. Using the results of [7] in both the Lagrangian and the Hamiltonian context, the isomorphism (6.13) means that

(i) the space of inequivalent Lagrangian conservation laws of the first group is isomorphic to the subspace of spatial local functionals in the coordinates and momenta, defined on the constraint surface $\tilde{\Sigma}$ and gauge invariant

18

on this surface, whose Poisson bracket with the first class Hamiltonian H_0 plus the explicit time derivative vanishes on the contraint surface,

$$\{Q = \int_{t=t_0} \tilde{b}_0 [\phi^i \widetilde{\pi_j}] \equiv \int_{t=t_0} d^1 x \dots d^{n-1} x \tilde{j}^0 [\phi^i \widetilde{\pi_j}],$$
$$[Q, H_0]_P + \frac{\partial}{\partial t} Q = 0|_{\tilde{\Sigma}}, Q \sim Q|_{\tilde{\Sigma}}\},$$

and

(ii) the space of inequivalent Lagrangian conservation laws of the second group is isomorphic to a subspace of the characteristic cohomology of the constraint surface in spatial form degree $(n-1)-1$, the space of conservation laws associated to the contraint surface, where two such conservation laws have to be considered to be equivalent if they differ on the constraint surface by a spatial superpotential and the total time derivative of a spatial current,

$$\{\tilde{j}^k, \partial_k \tilde{j}^k = 0|_{\tilde{\Sigma}}, \tilde{j}^k \sim|_{\tilde{\Sigma}} \tilde{j}^k + \partial_j \tilde{S}^{[jk]} + \frac{\partial}{\partial t} \tilde{f}^k - \{h_0, \tilde{f}^k\}_{P,alt}\}. \qquad (6.21)$$

For example, the current corresponding to the Lagrangian current $j^\mu = F^{0\mu}$ is given by the momenta π^k in the case of electromagnetism.

The semi-direct sum structure holds also for the Lagrangian Dickey algebra, but furthermore, we get from (6.19) that (i) the algebra of inequivalent conserved charges $\mathcal{Q}$ equipped with the induced Dickey bracket corresponds to the ordinary Poisson bracket algebra of conserved inequivalent charges in the Hamiltonian formalism and (ii) that the ideal I of conserved currents without charge forms an abelian subalgebra.

7 Conclusion

We have shown what is the precise relationship between the antibracket map and various Lie algebras existing for local gauge field theories. In the case of conserved currents, where "covariant" Poisson brackets are known, a direct comparision has been given.

In the general case, the antibracket map is related to the Poisson bracket of the canonical formalism. The core of this analysis is the relationship of the local BRST cohomologies in the Lagrangian and the Hamiltonian formalisms (i.e., the cohomologies modulo d in the Lagrangian case and modulo $\tilde{d}$ in the

19

Hamiltonian one). This relationship turns out to be somewhat more subtle than for the ordinary cohomologies, or the cohomologies modulo $\tilde{d}$, which are simply isomorphic.

We have shown in particular what is the precise analog of the Lie algebra of inequivalent conserved currents in the Hamiltonian framework, which in turn allows some general statements on the structure of this Lie algebra and could be useful for its actual computation.

Acknowledgements

Appendix A : Jet-spaces, variational bicomplex and Koszul-Tate resolution

In this appendix, we recall briefly the construction of jet-bundles and of the variational bicomplex. We will construct a tricomplex containing the horizontal, the vertical and the Koszul-Tate differentials. The construction enhances the cohomological setup of the variational bicomplex associated to possibly degenerate partial differential equations by implementing the pullback from the free bicomplex to the bicomplex of the surface defined by the equations through the homology of the Koszul-Tate differential. (Different considerations on the Batalin-Vilkovisky formalism in the context of the variational bicomplex are given in [14].)

Let us first recall some of the ingredients of the variational bicomplex relevant for our purpose (for a review see [8, 15, 16] and the references to the original literature therein). As we will not be concerned with global properties, we will work in local coordinates throughout. Consider a trivial fiber bundle

$$\pi : E = \mathbf{M}^n \times F \to \mathbf{M}^n \tag{A.1}$$

with local coordinates

$$\pi : (x^\mu, \phi^i) \to (x^\mu) \tag{A.2}$$

where $\mu = 0, \ldots, n - 1$ and $i = 1 \ldots, m$, with F a manifold homeomorphic to $\mathbf{R}^m$ parametrized by the ϕ^i. For simplicity, we assume here that all the ϕ^i are even, but all the considerations that follow could also be done in the case where the original bundle is a superbundle.

The induced coordinates on the infinite jet bundle

$$\pi^\infty : J^\infty(E) = \mathbf{M}^n \times F^\infty \to \mathbf{M}^n \tag{A.3}$$

of jets of sections on $\mathbf{M}^n$ are given by

$$(x^\mu, \phi^i, \phi^i_\mu, \phi^i_{\mu_1 \mu_2}, \ldots,). \tag{A.4}$$

Let $\Omega^p(J^\infty(E))$ be the local differential forms on $J^\infty(E)$. The exterior differential d_T is split into horizontal and vertical differentials: $d_T = d_H + d_V$, with

$$d_H = dx^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial L}{\partial x^\mu} + \phi^i_{(\nu)\mu} \frac{\partial L}{\partial \phi^i_{(\nu)}} \tag{A.5}$$

and

$$d_V \phi^i_{(\nu)} = d\phi^i_{(\nu)} - dx^\mu \phi^i_{(\nu)\mu}, \ d_V x^\mu = 0. \tag{A.6}$$

Note that everywhere else in the paper, we have omitted the subscript H on the horizontal differential and that we have introduced the more compact notation $\phi^i_{(\nu)} \equiv \partial_{(\nu)} \phi^i$ for the independent coordinates corresponding to the derivatives of the fields.

Furthermore, we have

$$d_H d_V + d_V d_H = d_H^2 = d_V^2 = 0. \tag{A.7}$$

A local p-form of $\Omega^p(J^\infty(E))$ can then be written as a sum of terms of the form $f[\phi]dx^{\mu_1} \ldots dx^{\mu_r} d_V \phi^{i_1}_{(\nu_1)} \ldots d_V \phi^{i_s}_{(\nu_s)}$ of horizontal degree r and vertical degree s with $r + s = p$ and $f[\phi]$ a smooth functions of x^μ, ϕ^i and a finite number of their derivatives. The free variational bicomplex is the double complex $(\Omega^{*,*}((J^\infty(E)), d_H, d_V)$ of differential forms on $(J^\infty(E))$.

$$
\begin{array}{ccccccccc}
& & \vdots & & & & \vdots & & \vdots \\
& & d_V \uparrow & & & & d_V \uparrow & & \delta_V \uparrow \\
0 & \longrightarrow & \Omega^{0,2}(J^\infty(E)) & \xrightarrow{\ d\ } & \cdots & \xrightarrow{\ d\ } & \Omega^{n,2}(J^\infty(E)) & \xrightarrow{\ \int\ } & \mathcal{F}^2(J^\infty(E)) & \longrightarrow & 0 \\
& & d_V \uparrow & & & & d_V \uparrow & & \delta_V \uparrow \\
0 & \longrightarrow & \Omega^{0,1}(J^\infty(E)) & \xrightarrow{\ d\ } & \cdots & \xrightarrow{\ d\ } & \Omega^{n,1}(J^\infty(E)) & \xrightarrow{\ \int\ } & \mathcal{F}^1(J^\infty(E)) & \longrightarrow & 0 \\
& & d_V \uparrow & & & & d_V \uparrow & & \delta_V \uparrow \\
0 \longrightarrow \mathbf{R} \longrightarrow & & \Omega^{0,0}(J^\infty(E)) & \xrightarrow{\ d\ } & \cdots & \xrightarrow{\ d\ } & \Omega^{n,0}(J^\infty(E)) & \xrightarrow{\ \int\ } & \mathcal{F}^0(J^\infty(E)) & \longrightarrow & 0 \\
& & \uparrow & & & & \uparrow & & \uparrow \\
0 \longrightarrow \mathbf{R} \longrightarrow & & \Lambda^0(\mathbf{M^n}) & \xrightarrow{\ d\ } & \cdots & \xrightarrow{\ d\ } & \Lambda^n(\mathbf{M^n}) & \longrightarrow & 0 \\
& & \uparrow & & & & \uparrow & & \\
& & 0 & & & & 0 & &
\end{array}
$$

The important property of this bicomplex is that all the rows and columns of the above diagram are exact [8, 15, 4]. The integral sign $\int$ denotes the projection, for each vertical degree s, of horizontal n-forms onto the space of local functional forms $\mathcal{F}^s$, i.e., the space of equivalence classes obtained by identifying exact horizontal n-forms with zero: $\mathcal{F}^s = \Omega^{n,s}/d_H\Omega^{n-1,s}$. δ_V is the induced action of the vertical derivative in $\mathcal{F}^s$: $\delta_V \int \omega^{n,s} = \int d_V \omega^{n,s}$. An evolutionary vector field on E is given by $v_Q = Q^i[\phi]\frac{\partial L}{\partial \phi^i}$. Its prolongation is given by $\delta_Q = \partial_{(\nu)}Q^i\frac{\partial L}{\partial \phi^i_{(\nu)}}$. Because $[\delta_Q, \partial_\mu] = 0$, the contraction of a functional form with the prolongation of evolutionary vector fields is well defined.

A system $\mathcal{R}$ of k-th order partial differential equations on E,

$$
\mathcal{R}_a\big(x^\mu, \phi^i, \phi^i_\mu, \ldots, \phi^i_{\mu_1\ldots\mu_k}\big) = 0 \qquad a = 1, \ldots, l, \tag{A.8}
$$

defines a subbundle $\mathcal{R} \to \mathbf{R}^n$ of $J^k(E) \to \mathbf{R}^n$. We shall assume that the equations $\mathcal{R}_a = 0, \partial_\mu\mathcal{R}_a = 0, \ldots, \partial_{\mu_1\ldots\mu_s}\mathcal{R}_a = 0$ define, for each x^μ, a smooth surface and provide a regular representation of this surface in the vector spaces F^{s+k} for each s, i.e., the equations can be split into independent equations (L_m) which can be locally taken as first coordinates in a new, regular, coordinate system in the vicinity of the surface defined by the equations, and into dependent ones (L_Δ) which hold as a consequence of the independent ones.

This implies that one can split locally the ϕ^i and their derivatives up to order $s + k$ into independent variables x_A not constrained by the equations and dependent variables z_α in such a way that the equations $\mathcal{R}_a = 0, \ldots, \partial_{\mu_1 \ldots \mu_s} \mathcal{R}_a = 0$ are equivalent to $z_\alpha = z_\alpha(x_A, L_m)$. A local coordinate system adapted to the equations is then given by (x^μ, x_A, L_m) in $J^{s+k}(E)$. How this works in detail for Yang-Mills theory, gravity or two-form fields, is discussed in [7]. The infinite prolongation $\mathcal{R}^\infty$ of $\mathcal{R}$, i.e., the given sets of equations and all their total derivatives, defines a subbundle in $J^\infty(E)$. In the sequel, by "stationary surface" or by "on-shell" we mean that we are on the subbundle defined by $\mathcal{R}_a = 0$ and an appropriate number of its derivatives, depending on the space $J^l(E)$ under consideration.

A consequence of the regularity condition is that any function $f[\phi]$ which vanishes on the stationary surface, $f \approx 0$, can be written as a combination of the equations defining this surface [8, 6].

The knowledge of the split of the equations into dependent and independent ones allows one to find a locally complete set of non trivial local reducibility operators in $J^l(E)$, i.e., operators $R_{a_1}^{+a(\mu)} \partial_{(\mu)}$ for some local functions $R_{a_1}^{+a(\mu)}[\phi]$ on $J^l(E)$ which do not all vanish on-shell such that

$$R^{+a(\mu)} \partial_{(\mu)} \mathcal{R}_a = 0 \tag{A.9}$$

and verifying the property that if $\lambda^{+a(\mu)} \partial_{(\mu)} \mathcal{R}_a = 0$ for some local functions $\lambda^{+a(\mu)}[\phi]$ on $J^l(E)$, then

$$\lambda^{+a(\rho)} \partial_{(\rho)} = \lambda^{+a_1(\lambda)} \partial_{(\lambda)} (R_{a_1}^{+a(\mu)} \partial_{(\mu)} \cdot) + \mu^{a(\mu)b(\nu)} (\partial_{(\nu)} \mathcal{R}_b) \partial_{(\mu)} \tag{A.10}$$

for some local functions $\lambda^{+a_1(\lambda)}[\phi]$, and $\mu^{a(\mu)b(\nu)}$ on $J^l(E)$, where $\mu^{a(\mu)b(\nu)} = -\mu^{b(\nu)a(\mu)}$. Furthermore, the first term of the right hand side of equation (A.10) can be assumed to be absent if the functions $\lambda^{+a(\mu)}$ vanish on-shell [6]. Such reducibility operators will be called trivial because they exist for any gauge theory.

For simplicity, we will assume here that the reducibility operators are themselves irreducible in the sense that if $\lambda^{+a_1(\lambda)} R^{+a(\mu)} \partial_{(\mu)}$ vanishes on the stationary surface, the functions $\lambda^{+a_1(\lambda)}$ vanish on the stationary surface. All the considerations that follow can be generalized to the case with higher order reducibility operators at the price of increasing the number of additional generators introduced below like in [6].

23

The variational bicomplex $(\Omega^{*,*}(\mathcal{R}^\infty), d_H, d_V)$ of the differential equations $\mathcal{R}$ is the pull-back of the variational bicomplex from $J^\infty(E)$ to $\mathcal{R}^\infty$. With the previous assumptions, it is straightforward to verify that $\Omega^{*,*}(\mathcal{R}^\infty)$ is locally isomorphic to the forms in dx^μ and $d_V x_A$ with coefficients that are smooth functions in the x^μ, x_A. The columns of this bicomplex remain exact, because the contracting homotopy [8] which allows to prove exactness in the free case still holds when we consider only $d_V x_A$'s. There exist however non trivial cohomology groups along the rows.

The Koszul-Tate resolution of this bicomplex is obtained by a straightforward generalization of the Koszul-Tate resolution of the stationary surface $\mathcal{R}^\infty$ [6]. One considers the superbundle

$$\pi : K = \mathbf{R} \times (F \oplus \Phi^* \oplus C^*) \longrightarrow \mathbf{R} \qquad (A.11)$$

and the associated free variational bicomplex $(\Omega^{*,*}(J^\infty(K)), d_H, d_V)$. Φ^* is the vector space with coordinates the Grassmann odd ϕ_a^* and is of dimension l, the number of original equations. C^* the vector space with coordinates the Grassmann even $C_{a_1}^*$ and its dimension equals the number of non-trivial reducibility operators $R_{a_1}^{+a(\mu)}\partial_{(\mu)}$. The Koszul-Tate differential δ is defined on $\Omega^{*,*}(J^\infty(K))$ by

$$\delta x^\mu = \delta dx^\mu = \delta\phi^i = 0,$$
$$\delta\phi_a^* = \mathcal{R}_a, \quad \delta C_{a_1}^* = R_{a_1}^{+a(\mu)}\partial_{(\mu)}\phi_a^*$$
$$\delta d_H + d_H\delta = 0 = \delta d_V + d_V\delta \qquad (A.12)$$

and is extended as a left antiderivation. The associated grading is obtained from the eigenvalues of the antighost number operator defined by

$$antigh = \phi_{a(\mu)}^* \frac{\partial^L}{\partial\phi_{a(\mu)}^*} + d_V\phi_{a(\mu)}^* \frac{\partial^L}{\partial d_V\phi_{a(\mu)}^*}$$
$$+2C_{a_1(\mu)}^* \frac{\partial^L}{\partial C_{a_1(\mu)}^*} + 2d_V C_{a_1(\mu)}^* \frac{\partial^L}{\partial d_V C_{a_1(\mu)}^*}. \qquad (A.13)$$

As in [6], one can then prove that

$$H_0(\delta, \Omega_0^{*,*}(J^\infty(K))) \simeq \Omega^{*,*}(J^\infty(E))/\mathcal{N} \simeq \Omega^{*,*}(\mathcal{R}^\infty) \qquad (A.14)$$

and that

$$H_k(\delta, \Omega_k^{*,*}(J^\infty(K))) = 0, \quad for\ k > 0. \qquad (A.15)$$

Here, $\mathcal{N}$ is the ideal of forms such that each term contains at least one of the terms $\partial_{(\mu)}\mathcal{R}_a$ or $d_V\partial_{(\mu)}\mathcal{R}_a$. Hence, locally, the quotient is isomorphic to the forms in dx^μ and $d_V x_A$ with coefficients that are smooth functions in the x^μ, x_A. By using a partition of unity, we then get the last isomorphism in the above equation. This means that the diagram

$$\ldots \xrightarrow{\delta} \Omega_k^{*,*}(J^\infty(K)) \xrightarrow{\delta} \Omega_{k-1}^{*,*}(J^\infty(K)) \xrightarrow{\delta} \ldots$$
$$\ldots \xrightarrow{\delta} \Omega_1^{*,*}(J^\infty(K)) \xrightarrow{\delta} \Omega^{*,*}(\mathcal{R}^\infty) \longrightarrow 0$$

is exact.

In the three dimensional grid corresponding to the tricomplex

$$(\Omega_*^{*,*}(J^\infty(K)), d_H, d_V, \delta)) \tag{A.16}$$

augmented by the projection on local functionals in the d_H direction and by the projection on the bicomplex for the partial differential equations $(\Omega^{*,*}(\mathcal{R}^\infty), d_H, d_V)$ in the δ direction, except for the rows of this last complex, the sequences are exact in all directions.

The advantage of this cohomological resolution of the variational bicomplex for partial differential equations is that the non trivial cohomology groups $H^{r,*}(d_H, \Omega^{r,*}(\mathcal{R}^\infty))$ are given by relative cohomology groups in the free tricomplex

$$H^{r,*}(d_H, \Omega^{r,*}(\mathcal{R}^\infty)) \simeq H_0^{r,*}(d_H|\delta, \Omega_0^{r,*}(J^\infty(K))). \tag{A.17}$$

Since $H_*^{q,*}(d_H, \Omega_*^{q,*}(J^\infty(K))) = 0$ for $0 < q < n$ and $H_k^{*,*}(\delta, \Omega_k^{*,*}(J^\infty(K))) = 0$ for $k > 0$, one can for instance apply the method of diagram chasing (or "snake lemma") in the horizontal and δ directions to get, for $(r, s) \neq (0,0)$,

$$H_0^{r,s}(d_H|\delta, \Omega_0^{r,s}(J^\infty(K))) \simeq H_1^{r+1,s}(d_H|\delta, \Omega_1^{r,s}(J^\infty(K))) \simeq \ldots$$
$$\simeq H_{n-r-1}^{n-1,s}(d_H|\delta, \Omega_{n-r-1}^{n-1,s}(J^\infty(K))). \tag{A.18}$$

For $(r, s) = (0, 0)$, the same chain of isomorphisms remain true if one replaces the first element in the chain by $H^{0,0}(d_H|\delta, \Omega_0^{0,0}(J^\infty(K)))/\mathbf{R}$. Furthermore, like in [10], one proves that:

$$\frac{H_k^{r,*}(d_H|\delta, \Omega_k^{r,*}(J^\infty(K)))}{p^\# H_k^{r,*}(d_H, \Omega_k^{r,*}(J^\infty(K)))} \simeq \frac{H_{k+1}^{r+1,*}(\delta|d_H, \Omega_{k+1}^{r+1,*}(J^\infty(K)))}{p^\# H_{k+1}^{r+1,*}(\delta, \Omega_{k+1}^{r+1,*}(J^\infty(K)))} \tag{A.19}$$

where $p^{\#}$ denotes the natural inclusion of an absolute cohomology group as a relative cohomology group. Using the results on the cohomology of d_H and δ, these relations reduce to

$$H_0^{0,0}(d_H|\delta, \Omega_0^{0,0}(J^\infty(K)))/\mathbf{R} \simeq H_1^{1,1}(\delta|d_H, \Omega_1^{1,1}(J^\infty(K))) \qquad \text{(A.20)}$$

$$H_k^{r,*}(d_H|\delta, \Omega_k^{r,*}(J^\infty(K))) \simeq H_{k+1}^{r+1,*}(\delta|d_H, \Omega_{k+1}^{r+1,*}(J^\infty(K))),$$
$$(r, s, k) \neq (0, 0, 0), r < n. \qquad \text{(A.21)}$$

Appendix B : Local brackets and surface terms

In the first part of this appendix, we want to calculate explicitly the total divergences that arise in the Jacobi identity for the local (anti)bracket.

Let $z^a = (\phi^A; \phi_A^*)$ and $\zeta^{ab} = \begin{pmatrix} 0 & \delta_B{}^A \\ -\delta_B{}^A & 0 \end{pmatrix}$. Let $(\tilde{\nu})_\mu$ denote the number of times the index μ appears in the multiindex (ν). The higher Euler operators [8] are uniquely defined by the expression

$$\delta_Q f = \partial_{(\nu)}(Q^a \frac{\delta^L f}{\delta z^a_{(\nu)}}). \qquad \text{(B.1)}$$

Let us furthermore define the "generalized Hamiltonian vector field" :

$$\bar{a}^b = (\frac{\delta^R \hat{a}}{\delta z^a})\zeta^{ab}. \qquad \text{(B.2)}$$

Then the local antibracket in the space of integrands (2.9) can be rewritten as

$$\{a_1, a_2\} = d^n x[\partial_{(\mu)}(\frac{\delta^R \hat{a}_1}{\delta z^a})\zeta^{ab}\frac{\partial^L \hat{a}_2}{\partial z^b_{(\mu)}})$$

$$-\frac{(\tilde{\nu})_\mu + 1}{|\nu| + 1}\partial_{\mu(\nu)}(\frac{\delta^R \hat{a}_1}{\delta z^a}\zeta^{ab}\frac{\delta \hat{a}_2}{\delta z^b_{\mu(\nu)}})] \equiv \delta_{\bar{a}_1} a_2 - dI_{\bar{a}_1} a_2. \qquad \text{(B.3)}$$

This expression implies that the graded Leibnitz rule holds up to a total divergence.

We have pointed out in the text that the local antibracket (B.3) does not satisfy the graded Jacobi identity strictly, but only up to a total divergence. Similarily, in the Hamiltonian theory, the Poisson bracket among local

functions of the fields, their conjugate momenta and their derivatives,

$$\{\hat{a}_1, \hat{a}_2\}_P = \frac{\tilde{\delta}^R \hat{a}_1}{\delta \phi^i} \sigma^{ij} \frac{\tilde{\delta}^L \hat{a}_2}{\delta \phi^j}, \tag{B.4}$$

satisfies the Jacobi identity $\{\hat{a}, \{\hat{b}, \hat{c}\}\} + cyclic = 0$ only up to a (spatial) total divergence. In equation (B.4), ϕ^i collectively denotes the fields and their conjugate momenta, the tilde superscript denotes spatial Euler-Lagrange derivatives and $\sigma^{ij} = \begin{pmatrix} 0 & \delta_j{}^i \\ -\delta_j{}^i & 0 \end{pmatrix}$.

For definitness, we shall evaluate here explicitly the boundary terms in the Jacobi identity in the Hamiltonian case and assume that the fields ϕ^i and the densities $\hat{a}, \hat{b}$, and $\hat{c}$ are all even. We will however not write explicitly the tilded superscript to indicate the spatial derivatives. The calculation for the local antibracket or the local extended Poisson bracket is simply a matter of taking care of the sign factors.

We will need the following lemma :

$$(-\partial)_{(\alpha)}(f \frac{\partial}{\partial \phi^i_{(\alpha)}} \partial_\beta g) = -(-\partial)_{(\alpha)}(\partial_\beta f \frac{\partial}{\partial \phi^i_{(\alpha)}} g). \tag{B.5}$$

The proof of this lemma follows from a straightforward extension of the proof of $\frac{\delta}{\delta \phi^i}(\partial_\beta g) = 0$ in [17].

A direct calculation, using the analog of (B.3) for the Poisson bracket and the fact that the Euler-Lagrange derivatives annihilate total divergences, yields

$$\{a, \{b, c\}\} + cyclic = \{a, \{b, c\}\} - \{b, \{a, c\}\} - \{\{a, b\}, c\} \tag{B.6}$$
$$= \delta_{\bar{a}} \delta_{\bar{b}} c - \delta_{\bar{b}} \delta_{\bar{a}} c - \delta_{\overline{\{a,b\}}} c$$
$$-dI_{\bar{a}}(\delta_{\bar{b}} c) + dI_{\bar{b}}(\delta_{\bar{a}} c) + dI_{\overline{\{a,b\}}} c \tag{B.7}$$

We have

$$(\delta_{\bar{a}} \delta_{\bar{b}} - \delta_{\bar{b}} \delta_{\bar{a}}) c = \delta_{\bar{d}} c \tag{B.8}$$

with

$$\bar{d}^i \equiv (\delta_{\bar{a}}(\frac{\delta \hat{b}}{\delta \phi^j}) \sigma^{ji} - (\hat{a} \leftrightarrow \hat{b})$$

27

$$= \frac{\delta}{\delta\phi^j}(\delta_{\bar{a}}\hat{b})\sigma^{ji} - (-\partial)_{(\alpha)}[\frac{\partial}{\partial\phi^j_{(\alpha)}}(\partial_{(\beta)}\frac{\delta\hat{a}}{\delta\phi^k})\sigma^{kl}\frac{\partial\hat{b}}{\partial\phi^l_{(\beta)}}]\sigma^{ji} - (\hat{a} \leftrightarrow \hat{b})$$

$$= \frac{\delta}{\delta\phi^j}\{\hat{a},\hat{b}\}\sigma^{ji} - (-\partial)_{(\alpha)}[\frac{\partial}{\partial\phi^j_{(\alpha)}}(\frac{\delta\hat{a}}{\delta\phi^k})\sigma^{kl}\frac{\delta\hat{b}}{\delta\phi^l}]\sigma^{ji} - (\hat{a} \leftrightarrow \hat{b})$$

$$= 2\frac{\delta}{\delta\phi^j}\{\hat{a},\hat{b}\}\sigma^{ji} - (-\partial)_{(\alpha)}[\frac{\partial}{\partial\phi^j_{(\alpha)}}(\frac{\delta\hat{a}}{\delta\phi^k}\sigma^{kl}\frac{\delta\hat{b}}{\delta\phi^l})]\sigma^{ji}$$

$$= \frac{\delta}{\delta\phi^j}\{\hat{a},\hat{b}\}\sigma^{ji} = \{\bar{a,b}\}^i \quad \text{(B.9)}$$

To get the line before last, we have used repeatedly the abovementioned lemma (B.5). Hence,

$$\{a,\{b,c\}\} + cyclic = d(-I_{\bar{a}}(\delta_{\bar{b}}c) + I_{\bar{b}}(\delta_{\bar{a}}c) + I_{\{\bar{a,b}\}}c). \quad \text{(B.10)}$$

This is the desired formula.[9]□

We now prove that the expressions in equation (3.10) for the Dickey bracket are equivalent to the definition in equation (3.9). Let us write terms which vanish on-shell by $\delta()$. By applying the lemma (B.5), we find that, if X is the characteristic of a variational symmetry, the following equation holds :

$$\delta_X(\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^i})Y^i d^n x = -(-\partial)_{(\mu)}[\frac{\partial}{\partial\phi^i_{(\mu)}}(\partial_{(\nu)}X^j)\frac{\partial\hat{\mathcal{L}}_0}{\partial\phi^j_{(\nu)}}]Y^i d^n x =$$

$$-(-\partial)_{(\mu)}[\frac{\partial}{\partial\phi^i_{(\mu)}}(X^j)\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^j}]Y^i d^n x = -\delta_Y(X^j)\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^j}d^n x + d\delta() \quad \text{(B.11)}$$

Let us evaluate $d(-\delta_{X_1}j_2)$. Using (B.11) twice, we get,

$$d(-\delta_{X_1}j_2) = -\delta_{X_1}(\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^i}X_2^i)d^n x + d\delta() =$$

$$-\delta_{X_1}(X_2^i)\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^i}d^n x + \delta_{X_2}(X_1^j)\frac{\delta\hat{\mathcal{L}}_0}{\delta\phi^i}d^n x + d\delta() \quad \text{(B.12)}$$

$$= d(\delta_{X_2}j_1) + d\delta(). \quad \text{(B.13)}$$

---

[9]It also follows from this proof that the alternative bracket given by $\{a,b\}_{alt} = \delta_{\bar{a}}b$ satisfies a strict Jacobi identity under Leibnitz form (defined by the right hand side of (B.6)), using furthermore the fact that $\{\bar{a,b}\}_{alt} = \{\bar{a,b}\}$.

From this equation it also follows immediately that

$$d(-\delta_{X_1} j_2) = \frac{1}{2} d(\delta_{X_2} j_1) - d(\delta_{X_1} j_2) + d\delta(). \tag{B.14}$$

Using the triviality of the cohomology of d in form degree $n-1$ (>0) implies the first two expressions in equation (3.10).

From equation (B.12), it follows that

$$d(-\delta_{X_1} j_2) = [\delta_{[X_1,X_2]} \hat{\mathcal{L}}_0$$
$$-\frac{(\tilde{\nu})_\mu + 1}{|\nu| + 1} \partial_{\mu(\nu)} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}} [X_1, X_2^i]) d^n x + d\delta(). \tag{B.15}$$

But we also have

$$\delta_{[X_1,X_2]} \hat{\mathcal{L}}_0 = (\delta_{X_2} \delta_{X_1} - \delta_{X_1} \delta_{X_2}) \hat{\mathcal{L}}_0$$
$$= \partial_\mu (\delta_{X_2} j_1^\mu - \delta_{X_1} j_2^\mu) +$$
$$\frac{(\tilde{\nu})_\mu + 1}{|\nu| + 1} \partial_{\mu(\nu)} [\delta_{X_2} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}} X_1^i) - \delta_{X_1} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}} X_2^i)] \tag{B.16}$$

This implies

$$d(-\delta_{X_1} j_2) = d(\delta_{X_2} j_1 - \delta_{X_1} j_2)$$
$$+\frac{(\tilde{\nu})_\mu + 1}{|\nu| + 1} \partial_{\mu(\nu)} (\delta_{X_2} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}}) X_1^i - \delta_{X_1} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}}) X_2^i) d^n x + d\delta(). \tag{B.17}$$

Using (B.14), we find the last expression of (3.10) :

$$d(-\delta_{X_1} j_2) = -\frac{(\tilde{\nu})_\mu + 1}{|\nu| + 1} \partial_{\mu(\nu)} (\delta_{X_2} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}}) X_1^i$$
$$-\delta_{X_1} (\frac{\delta \hat{\mathcal{L}}_0}{\delta \phi^i_{(\nu)\mu}}) X_2^i) d^n x + d\delta(). \square \tag{B.18}$$

In the last part of the appendix, we establish the relationship between the antibracket map and the Dickey bracket. As explained before theorem

29

4.2, we have to evaluate $\delta\{a_1, a_2\}$, where $a = d^n x \phi_i^* X^i$ with X^i defining a variational symmetry :

$$\delta\{a_1, a_2\} = (d^n x \frac{\delta X_1^i}{\delta \phi^j} X_2^j - \frac{\delta X_2^i}{\delta \phi^j} X_1^j) \frac{\delta \mathcal{L}_0}{\delta \phi^i}$$

$$= d^n x (\delta_{X_2} X_1^i - \delta_{X_1} X_2^i) \frac{\delta \mathcal{L}_0}{\delta \phi^i} + d\delta() = d(-\delta_{X_1} j_2) + d\delta(), \qquad \text{(B.19)}$$

where we have used (B.12) in order to get the last equality. This proves that to the antibracket map of two classes in $H_1^n(\delta|d)$ corresponds the Dickey bracket of the corresponding currents.□

Appendix C : Descent equations in the Hamiltonian formalism

We analyze in this appendix first of all the relationship between the cohomology of s_H defined in equation (6.3) and the cohomology of s_ω, thereby proving theorem 6.1. Then we analyze the spatio-temporal descent equations of s_H by choosing representatives appropriate to the Hamiltonian formalism, proving equations (6.7)-(6.12).

Cohomology of s_H and s_ω

The cocyle n in $s_H n = 0$ depends on the coordinates $x^\mu, \partial_{(\mu)}\tilde{\phi}^A, \partial_{(\mu)}\tilde{\phi}_A^*$. Consider the change of coordinates which consists in replacing the time derivatives of the fields and all their derivatives by the $\partial_{(\mu)}\mathcal{L}_A$. In the new coordinates n depends on $x^\mu, \partial_{(k)}\tilde{\phi}^A, \partial_{(\mu)}\tilde{\phi}_A^*, \partial_{(\mu)}\mathcal{L}_A$. Using $s_\omega \Omega = s_\omega H = 0$ and the identitiy

$$-s_\omega \{\tilde{\phi}_A^* \tilde{\phi}^A, \cdot\}_{P,alt} + \partial_{(k)} \frac{\tilde{\delta}^R}{\tilde{\delta}\tilde{\phi}^C}(\sigma^{CA}\{\tilde{\phi}_B^* \tilde{\phi}^B, \omega\}_{P,alt}) \frac{\partial^L}{\partial(\partial_{(k)}\tilde{\phi}^A)}$$
$$= \{\tilde{\phi}_A^* \tilde{\phi}^A, s_\omega \cdot\}_{P,alt}, \qquad \text{(C.1)}$$

we find that $s_H \mathcal{L}_A = 0$. This means that in the new coordinate system

$$s_H = s_\omega + \partial_{(\mu)}\mathcal{L}_A \frac{\partial^L}{\partial(\partial_{(\mu)}\tilde{\phi}_A^*)}, \qquad \text{(C.2)}$$

where s_ω is restricted to spatial derivatives. Introducing the contracting homotopy

$$\rho = \partial_{(\mu)}\tilde{\phi}_A^* \frac{\partial^L}{\partial(\partial_{(\mu)}\mathcal{L}_A)}, \tag{C.3}$$

the anticommutator $\{s_H, \rho\} = N = z^\alpha \frac{\partial^L}{\partial z^\alpha}$ is the operator counting the number of coordinates $z^\alpha \equiv \partial_{(\mu)}\tilde{\phi}_A^*, \partial_{(\mu)}\mathcal{L}_A$. The standard argument is then that

$$n = n(z^\alpha = 0) + \int_0^1 \frac{d\lambda}{\lambda}(Nn)[\lambda z^\alpha] \tag{C.4}$$

$$= n_0 + s_H(\int_0^1 \frac{d\lambda}{\lambda}(\rho n)[\lambda z^\alpha]. \tag{C.5}$$

The cocycle condition now reduces to $s_\omega n_0 = 0$ and the coboundary condition $n_0 = s_H p$ reduces to $n_0 = s_\omega p_0$. Indeed, applying N to the coboundary condition implies that $N s_H p = 0$. Using

$$[N, s_H] = 0 \tag{C.6}$$

and the same decomposition of p as for n in (C.4), this equation implies that $s_H b = s_\omega b_0$.

This proves theorem 6.1. $\Box$

In order to analyze the spatio-temporal descent equations for s_H, we start form the bottom, which we can assume to be of the form n_0 as above. We then want to know under what conditions n_0 can be lifted, i.e., what are the conditions for the existence of m such that $sm + dn_0 = 0$. We will now prove in particular the crucial lemma that m can be assumed to be independent of the coordinates $\partial_{(\mu)}\mathcal{L}_A$, with a linear dependence in the antifields $\partial_{(k)}\tilde{\phi}_A^*$ only in the terms involving the differential dt.

First lift from the bottom of the descent equations

The spatial exterior differential has the same form in the new coordinate system as it had in the old one. The total time derivative, however, is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \partial_{(k)l+1}\tilde{\phi}_A^* \frac{\partial^L}{\partial(\partial_{(k)l}\tilde{\phi}_A^*)} + \partial_{(k)l+1}\mathcal{L}_A \frac{\partial^L}{\partial(\partial_{(k)l}\mathcal{L}_A)} +$$
$$\partial_{(k)}\sigma^{CA}[-\mathcal{L}_C - \frac{\tilde{\delta}^R h}{\delta\tilde{\phi}^C} - \frac{\tilde{\delta}^R}{\delta\tilde{\phi}^C}\{\tilde{\phi}_B^*\tilde{\phi}^B, \omega\}_{P,alt}]\frac{\partial^L}{\partial(\partial_{(k)}\tilde{\phi}^A)}. \tag{C.7}$$

We then decompose m and n_0 into pieces respectively containing the differential dt or not (m^0, n_0^0 and $\tilde{m}, \tilde{n}_0$). The cocycle condition splits into :

$$s_H\tilde{m} + \tilde{d}\tilde{n}_0 = 0, \ \ s_H m^0 + \tilde{d}n_0^0 - \frac{d}{dt}\tilde{n}_0 = 0 \tag{C.8}$$

From the homotopy formula (C.4) applied to $\tilde{m}$ and the cocycle condition, we get that $\tilde{m} = \tilde{m}_0 + s_H() - \tilde{d}(\int_0^1 \frac{d\lambda}{\lambda}(\rho\tilde{n}_0)[\lambda z^\alpha])$ because ρ (anti)commutes with $\tilde{d}$ and $\tilde{d}$ is homogeneous of degree 0 in z^α. The last expression vanishes since $\rho\tilde{n}_0 = 0$. Injecting the remaining expression into the cocycle condition, we get

$$s_\omega\tilde{m}_0 + \tilde{d}\tilde{n}_0 = 0. \tag{C.9}$$

The homotopy formula (C.4) applied to m^0, together with the cocycle condition, implies that

$$m^0 = m_0^0 + s_H() - \tilde{d}(\int_0^1 \frac{d\lambda}{\lambda}(\rho n_0^0) + \int_0^1 \frac{d\lambda}{\lambda}\rho$$

$$(\sigma^{CA}\partial_{(k)}(-\mathcal{L}_C - \frac{\tilde{\delta}^R h}{\delta\tilde{\phi}^C} - \frac{\tilde{\delta}^R}{\delta\tilde{\phi}^C}\{\tilde{\phi}_B^*\tilde{\phi}^B, \omega\}_{P,alt})\frac{\partial^L}{\partial(\partial_{(k)}\tilde{\phi}^A)}\tilde{n}_0)[\lambda z^\alpha]$$

$$= m_0^0 + s_H() - \{\tilde{\phi}_A^*\tilde{\phi}^A, \tilde{n}_0\}_{P,alt}, \tag{C.10}$$

proving in particular the lemma on the dependence of m on the coordinates z^α. Injecting this last expression in the cocyle condition, using $s_\omega n_0^0 = 0$ and (C.1), implies

$$s_\omega m_0^0 + \tilde{d}n_0^0 - \frac{\partial}{\partial t}\tilde{n}_0 + \{h, \tilde{n}_0\}_{P,alt} = 0. \tag{C.11}$$

Next steps in the lifting procedure

We then have to try to lift the equivalent representative of m given by

$$m' = dt(-\{\tilde{\phi}_A^*\tilde{\phi}^A, \tilde{n}_0\}_{P,alt} + m_0^0) + \tilde{m}_0, \tag{C.12}$$

i.e., find $l = dt l^0 + \tilde{l}$ such that $s_H l + dm' = 0$. This implies

$$s_H\tilde{l} + \tilde{d}\tilde{m}_0 = 0, \ \ s_H l^0 + \tilde{d}(-\{\tilde{\phi}_A^*\tilde{\phi}^A, \tilde{n}_0\}_{P,alt} + m_0^0) - \frac{d}{dt}\tilde{m}_0 = 0. \tag{C.13}$$

32

By exactly the same reasoning as before, the first equation implies that

$$\tilde{l} = \tilde{l}_0 + s_H(), \ \ s_\omega \tilde{l}_0 + \tilde{d}\tilde{m}_0 = 0. \tag{C.14}$$

The second equation implies as before that

$$l^0 = l^0_0 + s_H() - \{\tilde{\phi}^*_A \tilde{\phi}^A, \tilde{m}_0\}_{P,alt} \tag{C.15}$$

because ρ annihilates the supplementary $\tilde{\phi}^*_A$-dependent term which does not depend on $\mathcal{L}_A$. Injecting into the cocycle condition, we get

$$s_\omega l^0_0 + \tilde{d}m^0_0 - \frac{\partial}{\partial t}\tilde{m}_0 + \{h, \tilde{m}_0\}_{P,alt} = 0, \tag{C.16}$$

the supplementary antifield dependent term in m^0 cancelling the term coming from (C.1) using the fact that $s_\omega \tilde{m}_0 + \tilde{d}\tilde{n}_0 = 0$. This shows that at every step, we get the same dependence on the coordinates z^α, i.e, independence on $\partial_{(\mu)}\mathcal{L}_A$, or by going back to the old coordinate system, on the time derivatives of the fields, with a linear dependence in the antifields and their spatial derivatives $\partial_{(k)}\tilde{\phi}^*_A$ only in the terms involving the differential dt. Furthermore, we have proved the set of equations (6.7)-(6.9).

Coboundary condition

Let us now consider the coboundary condition for l' defined in an analoguous way as m' in (C.12). From $l' = s_H r + du$, we have, by applying s_H, that $s_H u + dp = 0$. Hence u satisfies the same equation than l above, which implies by (C.14) and an appropriate modification of r, that we can assume $\tilde{u} = \tilde{u}_0$ and $u^0 = u^0_0 - \{\tilde{\phi}^*_A \tilde{\phi}^A, \tilde{p}_0\}_{P,alt}$.

We have that $\tilde{l}_0 = s_H \tilde{r} + \tilde{d}\tilde{u}$, which implies, by applying the homotopy formula (C.4) to r that we can assume that $\tilde{r} = \tilde{r}_0$, $\tilde{u} = \tilde{u}_0$. The coboundary condition becomes $\tilde{l}_0 = s_\omega \tilde{r}_0 + \tilde{d}\tilde{u}_0$, proving equation (6.11).

By applying the homotopy formula (C.4) to r^0, the coboundary condition

$$-\{\tilde{\phi}^*_A \tilde{\phi}^A, \tilde{m}_0\}_{P,alt} + l^0_0 = -s_H r^0 - \tilde{d}u^0 + \frac{d}{dt}\tilde{u}_0, \tag{C.17}$$

implies

$$-\{\tilde{\phi}^*_A \tilde{\phi}^A, \tilde{n}_0\}_{P,alt} + l^0_0 = -s_\omega r^0_0 - s_H \int_0^1 \frac{d\lambda}{\lambda}(\rho(+\{\tilde{\phi}^*_A \tilde{\phi}^A, \tilde{m}_0\}_{P,alt}$$
$$-l^0_0 - \tilde{d}u^0 + \frac{d}{dt}\tilde{u}_0))[\lambda z^\alpha]) - \tilde{d}u^0 + \frac{d}{dt}\tilde{u}_0. \tag{C.18}$$

This gives

$$l_0^0 = -s_\omega r_0^0 - \tilde{d}u_0^0 + \frac{\partial}{\partial t}\tilde{u}_0 - \{h, \tilde{u}_0\}_{P,alt}, \qquad (C.19)$$

proving equation (6.12). These coboundary conditions are satisfied by choosing in the equation $l = s_H r + du$, r to be given by $dt(-\{\tilde{\phi}_A^* \tilde{\phi}^A, \tilde{u}_0\}_{P,alt} + r_0^0) + \tilde{r}_0$ and a similar equation holding for u. This proves (6.10) $\square$.

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